1) Consider a mass spring system, with mass M and spring constant k. We put a mass m on top of the mass M. The coefficient of static friction between masses is μ_s, and there is no friction between the mass M and the ground.

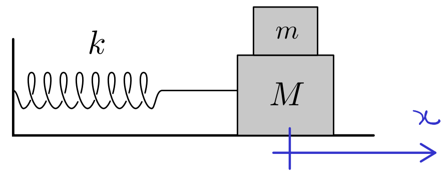


Figure 1: The masses held together through oscillations by friction.

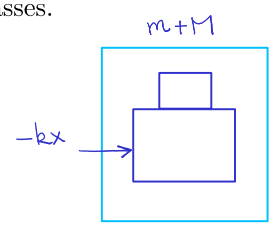
a) What is the frequency of the oscillations if the masses do not move relative to each other?
b) Find the maximum amplitude of oscillations if the masses do not move relative to each other.
c) [fun] If the coefficient of the kinetic friction, between the masses, is μ_k, and the amplitude is a bit larger than the maximum in part 'b' (the mass m stays on top of M), explain the motion of the masses.

a) $-kx = (m+M)\ddot{x}$

$\ddot{x} + \omega_0^2 x = 0$

$x(t) = A\cos(\omega_0 t + \varphi)$

$\omega_0 = \sqrt{\frac{k}{m+M}}$.

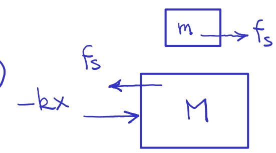


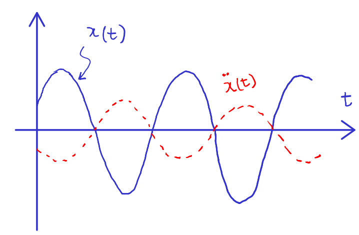
b) $f_s = m\ddot{x}$

$= -m\omega_0^2 A\cos(\omega_0 t + \varphi)$

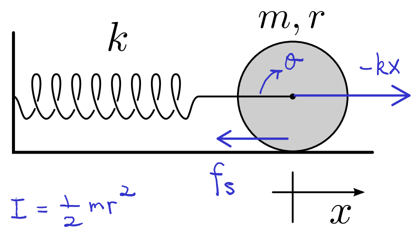
$f_{s\,max} = m\omega_0^2 A \le \mu_s mg$

$A_{max} = \frac{\mu_s g}{\omega_0^2}$.





2) Consider a uniform cylinder, with mass m and radius r, rolling without sliding. The axis of this cylinder is connected to a horizontal spring with spring constant k. See fig. 2. The whole motion can be explained with one degree of freedom, x, the movement from equilibrium point, and we choose right to be positive direction.
a) Write down the total kinetic energy of this system in terms of $\dot{x}$.
b) Write down the potential energy in terms of x.
c) Find the frequency of the oscillation.
d) If the static friction is finite, with coefficient of μ_s, find the maximum amplitude of the oscillation before any slipping happens.



rolling: $x = r\theta$, $\dot{x} = r\dot{\theta}$, $\ddot{x} = r\ddot{\theta}$

$I = \frac{1}{2} m r^2$

Figure 2: The oscillation with rolling.

a) $K = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2}\left(m + \frac{I}{r^2}\right)\dot{x}^2$ $\qquad \omega_0 = \sqrt{\frac{K_{eff}}{M_{eff}}}$

b) $U = \frac{1}{2} k x^2$

c) $E = \frac{1}{2}\left(m + \frac{I}{r^2}\right)\dot{x}^2 + \frac{1}{2} k x^2$ $\qquad \omega_0 = \sqrt{\frac{k}{m + I/r^2}}$

$\frac{dE}{dt} = 0 \rightarrow \ddot{x} + \omega_0^2 x = 0$

$x(t) = A\cos(\omega_0 t + \varphi)$

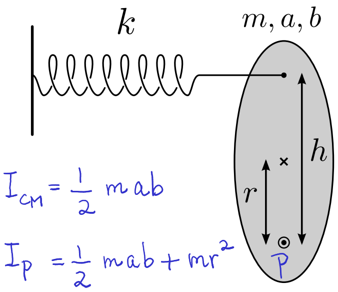
d) $-kx - f_s = m\ddot{x}$

$f_s r = I\ddot{\theta} = I\frac{\ddot{x}}{r}$

$f_s = \frac{I}{r^2}\ddot{x} = -\frac{I}{r^2} A\omega_0^2 \cos(\omega_0 t + \varphi)$

$f_s \le \mu_s m g \qquad A_{max} = \frac{\mu_s m g r^2}{I\omega_0^2}$.

3) A uniform elliptical disk, with mass m and axes lengths a and b, is pivoted on a point on its major axis, at a distance r from its center. The major axis is vertical and pivot is on the bottom. See fig. 3. We connect an spring with constant k to another point on the major axis, in a distance h from the pivot. Find the frequency of the small oscillations and discuss the condition for stability.



$$I_{CM} = \frac{1}{2} m a b$$

$$I_p = \frac{1}{2} m a b + m r^2$$

Figure 3: The oscillations of an ellipse.

$$\begin{cases} \sin\theta \approx \theta \\ \cos\theta \approx 1 \end{cases}$$

$$\tau_p = I_p \ddot{\theta}$$

$$\tau_p \approx + m g r \cancel{\sin\theta}^{\approx \theta} - k x h \approx m g r \theta - k h^2 \theta = (m g r - k h^2)\theta$$

$$x = h \sin\theta \approx h\theta$$

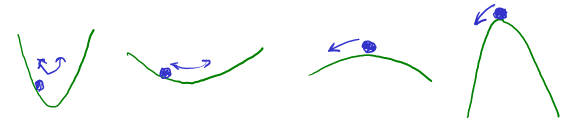
$$\ddot{\theta} + \frac{k h^2 - m g r}{I_p} \theta = 0$$

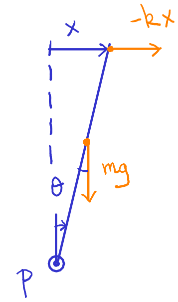
Sinh , Cosh

$$\omega_0 = \sqrt{\frac{k h^2 - m g r}{I_p}}$$

$$\underbrace{k h^2 - m g r > 0} \qquad \underbrace{k h^2 - m g r < 0}$$

pot. Energy





4) Consider a pulley, with mass m, radius r, and moment of inertia around its center I, which is held by a string passing around it and connected to the ceiling. This string at one end is connected to a spring with constant k and the other end is connected to the ceiling. See fig. 4.

a) Find the frequency of the small oscillations of this pulley. Assume that the string does not slide on the pulley.

b) [fun] If the coefficient of static friction between the string and the pulley is μ_s, find the maximum amplitude of the oscillations without any sliding happens.

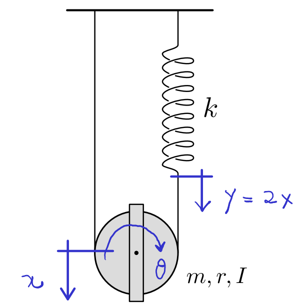
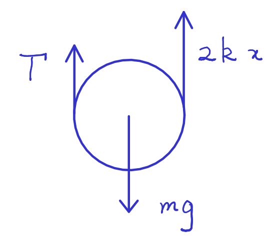


Figure 4: The oscillations of a pulley.

a) string length is const't. : $x - y + x = 0 \rightarrow y = 2x$.

not sliding Condition : $x = r\theta$



5) Consider two wheels rotating in opposite directions with constant angular velocities. The wheel on the right rotates counterclockwise. Their axes are separated by a distance l horizontally. We put a long rod with mass m on top of these wheels. The rod is sliding on the wheels with the coefficient of kinetic friction being μ_{k}.

a) Show that the rod will have a simple harmonic motion, and find the oscillation frequency.

b) If the angular velocity of the wheels are $\pm\Omega$, discuss the heat produced by the system.

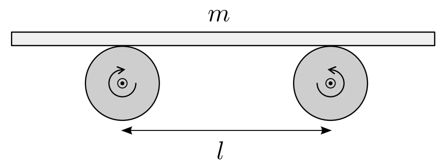


Figure 5: A rod oscillating between the wheels.

6) Discuss the equivalent spring constants for the springs connected in parallel or series. Say we have two springs with constants k_1 and k_2.
a) Find the effective spring constant when we connect these springs in parallel.
b) Fins the effective spring constant when we connect these springs in series.
c) Now, find the frequency of oscillations for the mass spring system shown in fig. 6. Other than the mass m everything else is massless.

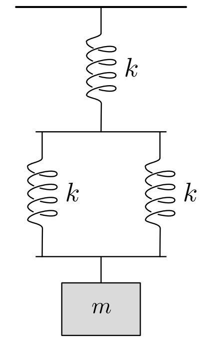


Figure 6: The springs system.

7) In each of the systems shown in the fig. 7, the properties of each system part is written close to the part. The mass is shown with parameters m or M, the moment of inertia about the center with I, the heights with h, the lengths with l or a or b, radii with r or R. All the masses are uniform. There is no sliding in any one of the systems. Find the frequency of small oscillations for each one of the systems.

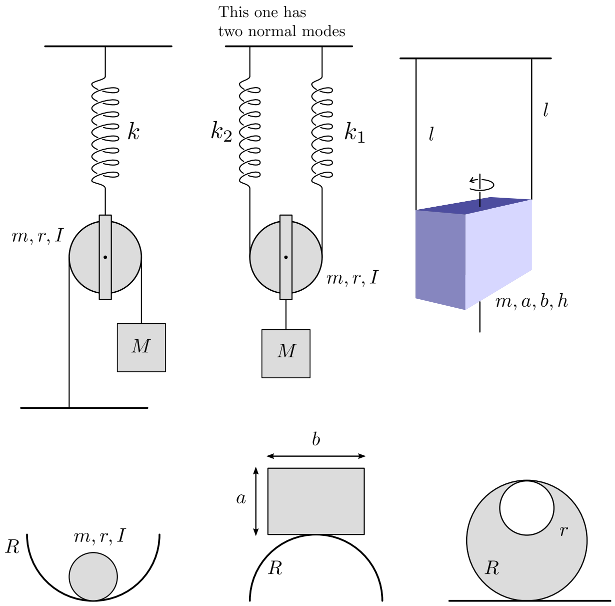


Figure 7: Find frequency of small oscillations in each one of these systems.