gravitation, problem set #2

1) We know the radius of the earth, $R_{\mathrm{E}} = 6.4 \times 10^3$ km, and the gravitational field on the surface, $g = 9.8$ m/s^2. Also we know moon goes around the earth once in $T = 29$ days and its orbit is almost a circle.
a) Find the mass of the earth.
b) How far is the moon from the earth?
c) We see the moon on the sky with $\theta = 0.5°$ angular diameter. Use previous part to find moon's radius.
d) Find moon's velocity relative to the earth.
e) With more accurate observation, we see that earth is rotating around the center of mass of the earth-moon system which is located at $R_{\mathrm{CM}} = 4.7 \times 10^3$ km from center of the earth. Find mass of the moon.

a) $$g = \frac{GM_E}{R_E^2} = 9.8\ m/s^2 \Rightarrow M_E = \frac{gR_E^2}{G}$$

b) $$r^3 = GM\frac{T^2}{4\pi^2} \rightarrow r = \left[\frac{GMT^2}{4\pi^2}\right]^{1/3}$$

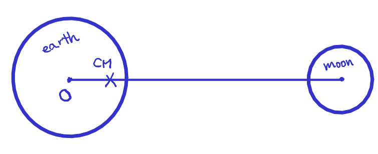
c) $$\tan\frac{\theta}{2} = \frac{R_{Moon}}{r}, \quad \theta \ll 1 \rightarrow \theta = \frac{2R_{Moon}}{r} \rightarrow R_{Moon} = \frac{r\theta}{2}$$

d) $$v = \frac{2\pi r}{T}$$

or

$$\frac{v^2}{r} = \frac{GM_E}{r^2} \Rightarrow v = \sqrt{\frac{GM_E}{r}}$$

e) $$R_{CM} = \frac{M_{Moon}\, r + M_E \times 0}{M_E + M_{Moon}}$$



$$= 4.7 \times 10^3\ km$$

$$\Rightarrow M_{Moon} = M_E \frac{R_{CM}}{r - R_{CM}}$$

2) Consider two planets with masses M_1 and M_2, and radii R_1 and R_2, sitting at $x_1 = 0$ and $x_2 = r$. For the first two parts assume they are not moving.

a) Find the point between these two masses that the gravitational field is zero and call it X.
b) We throw a stone from planet 2 to planet 1 from point P. What is the minimum initial velocity, $v_{\rm i}$, needed to do that? We only need to reach the point you found in the previous part.

Now we assume that $M_1 \gg M_2$, and M_2 is rotating around M_1 in a circular orbit. You can think about sun-earth or earth-moon for example, but there is no spin. It turned out that to throw a stone with minimum push, from planet two to planet one, you should throw it as shown in fig. 1, using Hohmann path. We throw the stone, with velocity v relative to M_2, in the direction that M_2 moves so it adds to the velocity. This path is an ellipse which you know the major axis. It touches the surface of planet 1 and stops.

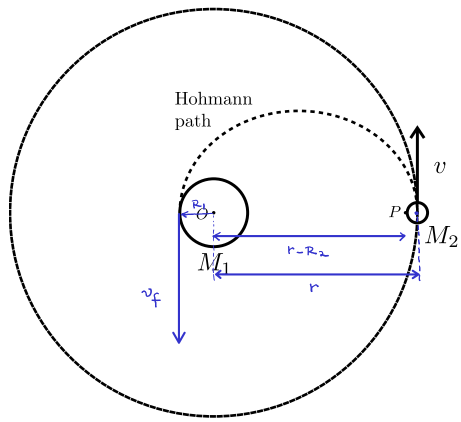
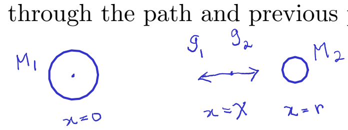


Figure 1: Throwing stone using Hohmann path.

c) What is the velocity of M_2 around M_1? Call this V, and define $v_{\rm i} = v + V$.
d) Angular momentum of the stone with respect to point O is conserved in this path. Write down an equation for this in terms of $v_{\rm i}, v_{\rm f}, r, R_1, R_2$.
e) How long it takes for the stone to travel this path? ($r \gg R_2$)
f) Using the fact that the energy of the stone conserved through the path and previous parts, find $v_{\rm i}$.

a) $g_1 = g_2 \Rightarrow \dfrac{GM_1}{X^2} = \dfrac{GM_2}{(r-X)^2}$



$X \neq 0, r$
$\Rightarrow M_1 (r-X) = M_2 X^2$

$\Rightarrow \sqrt{M_1}\,|r-X| = \sqrt{M_2}\,|X| \overset{0<X<r}{\Rightarrow} X = r\dfrac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}} = r\left[1+\sqrt{\dfrac{M_2}{M_1}}\right]^{-1}.$

b) at point $x = X$, $v = 0$, & the energy is conserved, (m is mass of the stone)

$E = -\dfrac{GM_1 m}{X} - \dfrac{GM_2 m}{r-X}$, right after throwing at P, $v = v_i$ and the stone is at the surface of M_2, so $E = \dfrac{1}{2}mv_i^2 - \dfrac{GM_1 m}{r-R_2} - \dfrac{GM_2 m}{R_2}$. So $v_i^2 = 2G\left[\dfrac{M_1}{r-R_2} - \dfrac{M_1}{X} + \dfrac{M_2}{R_2} - \dfrac{M_2}{r-X}\right].$

if $r, X \gg R_1, R_2$ then $v_i^2 = 2G\dfrac{M_2}{R_2}$, which is the escape velocity from M_2 if M_1 was not there.

c) $F = ma$: $\dfrac{GM_1M_2}{r^2} = M_2\dfrac{V^2}{r} \Rightarrow V = \sqrt{\dfrac{GM_1}{r}}.$

d) at the beginning, $L_i = mr\,v_i = mR_2(v+V)$, at the final point, $L_f = mR_1 v_f$, so $rv_i = R_1 v_f$.

e) for the ellipse, the major axis is, $2a = r + R_1$, and we need almost half the period ($r \gg R_2$), so $\dfrac{T}{2} = \dfrac{1}{2}\left[\dfrac{4\pi^2 a^3}{GM_1}\right]^{1/2} = \dfrac{\pi(r+R_1)^{3/2}}{2\sqrt{2GM}}.$

f) $E = \dfrac{1}{2}mv_i^2 - \dfrac{GM_2 m}{R_2} - \dfrac{GM_1 m}{r} = \dfrac{1}{2}mv_f^2 - \dfrac{GM_1 m}{R_1} - \dfrac{GM_2 m}{r+R_1}$ & $rv_i = R_1 v_f$, so $v_i = \left[\dfrac{2G\left(\dfrac{M_2}{R_2} - \dfrac{M_1}{r} + \dfrac{M_1}{R_1} - \dfrac{M_2}{r+R_1}\right)}{\left(\dfrac{r}{R_1}\right)^2 - 1}\right]^{1/2}$

and $v = v_i - V = v_i - \sqrt{\dfrac{GM_1}{r}}$, the velocity we need at the beginning.